Equilibrium and Stability of Normal and Superconducting Current Loops

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The conditions for equilibrium and stability of a normal or a superconducting current loop in the magnetic field of other fixed loops are derived from the energy principle, both with and without a gravitational field. While loops with finite resistance cannot be levitated, stable equilibrium arrangements of superconductors can be found. In this connection a sufficient criterion for the indefiniteness of homogeneous multilinear forms $\sum_{k_1 \ldots k_n} T_{k_1 \ldots k_n} X_{k_1} \ldots X_{k_n}$ was derived and applied.

Putting bodies which react to magnetic forces into a freely floating and stable equilibrium position (levitation) is a problem of interest in various branches of physics and engineering. Possible ways of achieving this have already been examined by various authors [1,2]. The present paper was induced by a problem of plasma physics.

In experiments with the levitron machine[3], a toroidal hard core tube for plasma confinement, a current carrying superconducting ring is dropped through the plasma. The annular current causes a large shear and rotational transform of the toroidal magnetic field, and in this way it is hoped to reduce plasma instabilities. It would be desirable to fix the ring during the experiment without any mechanical support. This paper examines the possibilities of achieving this by using magnetic forces.

In Section I the work is calculated which has to be done when a freely movable but solid current loop with finite resistivity is shifted or turned in the magnetic field of other current loops. All current loops are assumed to be negligibly thin compared with their longitudinal extensions. By making the resistivities approach zero or infinity the corresponding expressions for superconductors or loops with constant currents are obtained. In the following sections equilibrium and stability are examined, in Section II for loops with $\delta I = 0$, in Section III for superconductors and in Section IV for conductors with finite resistivity.

In Section II a sufficient criterion for the indefiniteness of homogeneous multilinear forms is deduced, which is applied when discussing stability.

I. Energy Balance

Let us take an ensemble of freely movable but inflexible conductors L_i with resistance R_i and current I_i. Their thickness should be so small compared with their longitudinal dimensions and mutual distances that they can be approximated by a one-dimensional model with self and mutual inductances defined solely by geometries. A somewhat more critical discussion of this point is given in the appendix.

The energy balance during mutual displacements of the loops is described by POYNTING's theorem

$$\dot{e} + \operatorname{div}(\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{j} \cdot \boldsymbol{E} = 0 .$$

e is the energy density of the electric and magnetic fields. When the displacement of the loops is slow enough, the radiative energy losses may be neglected and e can be set equal to the energy density e_B of the magnetic field. Integration over the whole space makes the second term vanish after changing it into a surface integral, and one obtains

$$\mathrm{d}/\mathrm{d}t \int e_B \, \mathrm{d}\tau + \int \boldsymbol{j} \cdot \boldsymbol{E} \, \mathrm{d}\tau = 0 .$$

After transforming the electric field in every loop to its value E^* in the restsystem we get in the limit of infinitely thin loops

$$\int \boldsymbol{j} \cdot \boldsymbol{E} \, \mathrm{d}\tau = \sum_i I_i \oint \boldsymbol{E} \, \mathrm{d}\boldsymbol{s}_i = \sum_i I_i \oint (\boldsymbol{E}^* - \boldsymbol{v}_i \times \boldsymbol{B}) \, \mathrm{d}\boldsymbol{s}_i ,$$

$\boldsymbol{v}_i$ being the velocity of L_i in the laboratory system, and I_i being the current flowing in L_i. Since the change of the magnetic flux Φ_i through the i-th

[1] R. D. WALDRON, Rev. Sci. Instr. **37**, 29 [1966], see further references there.

[2] V. N. BELOOZEROV, Sovjet Phys. — Tech. Phys. **11**, 631 [1966].

[3] D. H. BIRDSALL et al., Proc. Culham Conf. Sept. 1965, Vol. II, p. 291.

loop is given by

$$\oint \boldsymbol{E}^* \, \mathrm{d}\boldsymbol{s}_i = -\dot{\Phi}_i$$

there results

$$-\sum_i \boldsymbol{v}_i I_i \oint \mathrm{d}\boldsymbol{s}_i \times \boldsymbol{B} = \frac{\mathrm{d}}{\mathrm{d}t} \int e_B \, \mathrm{d}\tau - \sum I_i \dot{\Phi}_i .$$

$-I_i \oint (\mathrm{d}\boldsymbol{s}_i \times \boldsymbol{B})$ is the force which has to act when the i-th loop moves against the magnetic forces. Therefore, the left-hand side gives the work $\mathrm{d}W/\mathrm{d}t$ which must be done to move the loops.

Using the well-known relation

$$\int e_B \, \mathrm{d}\tau = \tfrac{1}{2} \sum_i I_i \Phi_i$$

gives

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \tfrac{1}{2} \sum_i (\dot{I}_i \Phi_i - I_i \dot{\Phi}_i). \tag{1}$$

The fluxes Φ_i are calculated from the currents I_i according to

$$\Phi_i = \sum_k L_{ik} I_k \tag{2}$$

where $L_{ik} = L_{ki}$ is the matrix of self and mutual inductances. The flux changes are subjected to FARADAY's law of induction

$$-\dot{\Phi}_i = R_i (I_i - I_i^{(0)}) \tag{3}$$

($I_i{}^{(0)}$ is the current in the loop L_i before any displacements were made). From (2) and (3) a system of differential equations results

$$\sum_k L_{ik} \dot{I}_k + \sum_k \dot{L}_{ik} I_k = -R_i (I_i - I_i^{(0)}) . \tag{4}$$

In the limit of superconductors only, i.e. $R_i \to 0$ for all i, it follows from (3) that all fluxes are constant and (1) can be integrated to give

$$W = \tfrac{1}{2} \sum_i I_i \Phi_i . \tag{5}$$

When all R_i are going to infinity, (4) shows that the currents remain constant and again (1) can be integrated:

$$W = -\tfrac{1}{2} \sum_i I_i \Phi_i . \tag{6}$$

If the weight of the loops is taken into account, on the right-hand sides of (5) and (6) the potential energy $V = \sum_i m_i \boldsymbol{g} \boldsymbol{x}_{is}$ has to be added. ($\boldsymbol{x}_{is}$ = coordinate vector of the centers of gravity.)

II. Instability for $\delta I_i = 0$ ($R_i \to \infty$)

In the limit of very high resistances R_i one gets from (6) using (2) and $\delta I_i = 0$

$$\delta^{(n)} W = -\tfrac{1}{2} \sum_{i,k} I_i I_k \delta^{(n)} L_{ik} \tag{7}$$

where $\delta^{(n)}$ is the n-th variation.

If one considers especially a pure shift $\boldsymbol{\xi}$ (without turn, see Fig. 1) of the loop L_0, while the position of the other loops $L_1, \ldots, L_n$ remains unchanged,

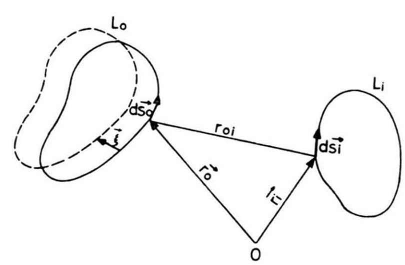

Fig. 1.

then only the inductances L_{0i}, $i = 1, \ldots, n$, will change, and because of

$$L_{0i} = \oint \oint \frac{\mathrm{d}\boldsymbol{s}_0 \cdot \mathrm{d}\boldsymbol{s}_i}{r_{0i}} \quad \text{one gets}$$

$$\begin{aligned} \delta^{(n)} L_{0i} &= \sum_{k_1 \ldots k_n = 1}^{3} \frac{1}{n!} \oint \oint \frac{\partial^n}{\partial \xi_{k_1} \ldots \partial \xi_{k_n}} \frac{1}{r_{0i}(\xi)} \cdot \mathrm{d}\boldsymbol{s}_0 \, \mathrm{d}\boldsymbol{s}_i \, \xi_{k_1} \ldots \xi_{k_n} \\ &=: \sum_{k_1 \ldots k_n} t^{0i}_{k_1 \ldots k_n} \xi_{k_1} \ldots \xi_{k_n} \end{aligned} \tag{8}$$

with $\quad r_{0i}(\boldsymbol{\xi}) = |\boldsymbol{r}_i + \boldsymbol{\xi} - \boldsymbol{r}_0|$.

Since

$$\delta^{(n)} L_{ik} = \delta^{(n)} L_{0i} \delta_{k0} + \delta_{0i} \delta^{(n)} L_{0k} \tag{9}$$

(δ_{ik} = KRONECKER's symbol), inserting (8) into (7) gives

$$\begin{aligned} \delta^{(n)} W &= -\frac{I_0}{n!} \sum_{i=1}^{n} I_i \sum_{k_1 \ldots k_n} t^{0i}_{k_1 \ldots k_n} \xi_{k_1} \ldots \xi_{k_n} \\ &=: \sum_{k_1 \ldots k_n} T_{k_1 \ldots k_n} \xi_{k_1} \ldots \xi_{k_n} . \end{aligned} \tag{10}$$

It may be shown now that any multilinear form

$T(\boldsymbol{\xi})$ of type (10), independent of a representation like (8), is nondefinite, if the contraction of any two indices, say k_i and k_l, vanishes. With momentary use of summation convention this condition is

$$T_{k_1..s..s..k_n} = 0 .$$

It then follows that

$$\Delta_\xi T = \frac{\partial}{\partial \xi_s}\frac{\partial}{\partial \xi_s} T_{k_1...k_n}\xi_{k_1}\cdots\xi_{k_n}$$
$$= T_{ssk_3...k_n}\xi_{k_3}\cdots\xi_{k_n} + \cdots + T_{k_1...k_{n-2}ss}\xi_{k_1}\cdots\xi_{k_{n-2}} = 0$$

i.e. T is a harmonic function of $\boldsymbol{\xi}$, and the well-known principle of non-existence of a maximum or minimum in the interior of a closed domain applies. Since $T(0)=0$ there is always at least one $\boldsymbol{\xi}$ with $T(\boldsymbol{\xi})>0$ and another $\boldsymbol{\xi}$ with $T(\boldsymbol{\xi})<0$, except $T\equiv 0$. (This proof is valid also for $n>3$ dimensions).

It remains to be shown that for the tensor $T_{k_1...k_n}$, defined by (8) and (10), the assumption of vanishing contractions is true, and this is most easily done using $\Delta(1/r)=0$ when $r\neq 0$ and the interchangeability of the differentiations in (8):

$$t^{0i}_{k_1..s..s..k_n} = \frac{1}{n!}\oint\oint \frac{\partial^n}{\partial\xi_{k_1}\dots\partial\xi_s\dots\partial\xi_s\dots\partial\xi_{k_n}}\frac{1}{r_{0i}(\xi)}\mathrm{d}\boldsymbol{s}_0\mathrm{d}\boldsymbol{s}_i$$
$$= \frac{1}{n!}\oint\oint\frac{\partial^{n-2}}{\partial\xi_{k_1}\dots\partial\xi_{k_n}}\frac{\partial^2}{\partial\xi_s\partial\xi_s}\frac{1}{r_{0i}(\xi)}\mathrm{d}\boldsymbol{s}_0\,\mathrm{d}\boldsymbol{s}_i = 0$$

and because

$$T_{k_1...k_n} = -\frac{1}{n!} I_0 \sum_i I_i t^{0i}_{k_1...k_n}$$

the assumption is immediately seen to be true.

Since it was found that all variations of W are indefinite or equal to zero it is impossible to realize the stability condition that the work done by any displacement $\boldsymbol{\xi}$ be positive. So there is no stable equilibrium under the condition $\delta I_i=0$. This statement is not altered by the addition of a homogeneous gravitational field, since the variations of the gravitational energy $\delta^{(n)}V=\delta^{(n)}\boldsymbol{g}\cdot\boldsymbol{\xi}$ are zero for $n\geqq 2$.

III. Stabilization of Superconducting Loops

In this section it is examined whether a superconducting loop L_0 can be held in a stable position by appropriate juxtaposition of other superconducting loops L_i, $i=1,\dots,n$, which are held in fixed positions, and this is done with and without a gravitational field.

Now Eq. (5) is valid, and adding a gravitational potential to the right hand side, gives, because of the constancy of the fluxes

$$\delta W = \tfrac{1}{2}\sum_i \Phi_i\,\delta I_i + \delta V,$$
$$\delta^{(2)} W = \tfrac{1}{2}\sum_i \Phi_i\,\delta^{(2)} I_i$$

since $\delta^{(2)}V = \delta^{(2)}\boldsymbol{g}\cdot\boldsymbol{\xi} = 0$. Using (2) and the relations following from it

$$\sum_i L_{ik}\,\delta I_i = -\sum_i I_i\,\delta L_{ik},$$
$$\sum L_{ik}\delta^{(2)} I_i = -\sum_i I_i\,\delta^{(2)} L_{ik} - 2\sum_i \delta I_i\,\delta L_{ik}$$

gives
$$\delta W = -\tfrac{1}{2}\sum_{i,k} I_i\, I_k\, \delta L_{ik} + \delta V \tag{11}$$

and
$$\delta^{(2)} W = \sum_{i,k} L_{ik}\delta I_i\delta I_k - \tfrac{1}{2}\sum_{i,k} I_i I_k \delta^{(2)} L_{ik}\,. \tag{12}$$

a) Equilibrium

The loop L_0 is in an equilibrium position when for any displacement out of this position no work has to be done in the lowest order, that is $\delta W=0$.

(11) together with (9) results in the equilibrium condition

$$-I_0\sum_{i=1}^{n} I_i\,\delta L_{i0} + \delta V = 0\,. \tag{13}$$

Any first order displacement of the rigid loop L_0 may be combined from a rectilinear displacement of the center of gravity and a rotation about it. Let δx^α, $\alpha=1,2,3$, be the rectilinear displacements along the axes of an orthogonal coordinate system, and δx^α, $\alpha=4,5,6$, the differential variations of the Eulerian angles. Then from (13) one gets

$$I_0\sum_{i=1}^{n} I_i\cdot\partial L_{i0}/\partial x^\alpha = m_0 g_\alpha,\quad \alpha=1,2,3, \tag{14}$$

$$I_0\sum_{i=1}^{n} I_i\cdot\partial L_{i0}/\partial x^\alpha = 0,\qquad \alpha=4,5,6 \tag{15}$$

where $m_0 g_\alpha$ stands for the space gradient $\partial V/\partial x^\alpha$, m_0 being the mass of the loop L_0 and g_α the component of the gravitational field $\boldsymbol{g}$ in the α-direction.

When the right-hand side of (14) is non-zero I_0 and at least one of the currents I_i in the "outer" loops has to be non-zero. According to (15) the angular gradients $\partial L_{i0}/\partial x^\alpha$, $\alpha=4,5,6$, have to be linearly dependent. When this is given, currents I_i such that (15) is satisfied can always be found. This could be very readily achieved by forming and arranging all loops with non-vanishing current in such a way that the L_{i0} do not change in rotations of L_0

about its center of gravity. ($\partial L_{i0}/\partial x^\alpha = 0$, $\alpha = 4, 5, 6$, $I_i \neq 0$). In order to satisfy Eq. (14) the entire arrangement of loops has to be rotated in such a way that the vector

$$\sum_{i=1}^{n} I_i \, \partial L_{i0}/\partial x^\alpha , \quad \alpha = 1, 2, 3 ,$$

assumes the direction of $\boldsymbol{g}$, and I_0 has to be chosen such that (14) is also quantitatively correct. Among the many realizations possible some choice is made by postulating stability according to the results in sub-section b) of this section.

In the special case of vanishing gravity the equilibrium condition can be fulfilled most easily by the choices $I_0 = 0$, or $I_0 \neq 0$, $I_i = 0$ for $i \geqq 1$, and for arbitrary shapes and arrangement of the loops.

A similarity condition which will be useful later on can be derived from (13). If I_0, $I_{i \geqq 1}$ give an equilibrium position the same is true of the currents $A\,I_0$, $A^{-1} I_{i\geqq 1}$ for the same configuration and for any $A \neq 0$. The current in L_0, in particular, can be made much higher than the currents in the outer loops.

b) Stability

For the equilibrium constructed in a) to be stable there has to be $\delta^{(2)} W > 0$ for all displacements of L_0. Since the second variation of the potential energy in a homogeneous gravitational field vanishes and since only the variations of the mutual inductances of L_0 are different from zero, using (12) gives the stability criterion

$$- I_0 \sum_{i=1}^{n} I_i \, \delta^{(2)} L_{i0} + \sum_{i,k=0}^{n} \delta I_i \, \delta I_k \, L_{ik} > 0 . \qquad (16)$$

In Section II it was shown that the first term of (16), which is identical with (7) for $n = 2$ is indefinite or vanishes altogether. On the other hand, the second term is always positive, except for all $\delta I_i = 0$, since the matrix L_{ik} is positive definite ($\frac{1}{2} \sum_{i,k} L_{ik} I_i I_k$ is the positive energy of the magnetic field).

For vanishing gravity in a) the most simple equilibrium solutions were found to be

$$I_0 = 0 \quad \text{or} \quad I_i = 0 , \quad i = 1, \ldots, n .$$

In both cases (16) is fulfilled, if the outer loops L_i, $i = 1, \ldots, n$, are arranged such that for any displacement of L_0 a current $\delta I_i \neq 0$ is induced in at least one of them.

This condition has to be satisfied also in the general case, if the sufficient (not necessary) stability criterion

$$\sum_{i,k=0}^{n} \delta I_i \, \delta I_k \, L_{ik} > \left| I_0 \sum_{i=1}^{n} I_i \delta^{(2)} L_{i0} \right| \qquad (17)$$

is substituted for (16). Since

$$\sum_k L_{ik} \, \delta I_k = - \sum_k I_k \, \delta L_{ik} \qquad (18)$$

is obtained from (2) because $\delta \Phi_i = 0$, it must also be true for any type of motion that $\sum_k \delta L_{ik} \, I_k \neq 0$ for one i at least. In other words, the system of equations

$$\begin{aligned} &\sum_k \delta L_{0k} \, I_k = 0 \\ &\delta L_{0i} I_0 = 0 , \quad i = 1, \ldots, n \end{aligned} \qquad (19)$$

should not have any solution. First it is obvious that there should not be any motion for which all δL_{0i} vanish simultaneously, because otherwise the first equation of the system (19) would also be satisfied along with the last equations. If, on the other hand, for every motion at least one δL_{0i} is different from zero, then at least one of the last n equations and hence the entire system will always fail to be satisfied if I_0 is different from zero. (This condition was obtained for equilibrium in the presence of a gravitational field and will be assumed during the whole of this section.) One thus obtains the condition that the system of equations

$$\sum_{\alpha=1}^{6} (\partial L_{i0}/\partial x^\alpha) \, \delta x^\alpha = 0 , \quad i = 1, \ldots, n \qquad (20)$$

should not have any solution, and hence the rank of the matrix $\partial L_{i0}/\partial x^\alpha$ must be equal to six, this being possible only when there are $n \geqq 6$ outer loops. If symmetry properties allow the number of degrees of freedom μ for the motion of L_0 to be reduced, the sum over α only goes to μ and $n \geqq \mu$ outer loops are sufficient for stabilization.

All that has been achieved so far is to make the left-hand side of (17) positive for all motions. It can be made larger than the right-hand side by making use of the similarity transformation proved at the end of a). According to (18) it holds for $i \geqq 1$ that

$$\sum_{k=0}^{n} L_{ik} \, \delta I_k = - \, \delta L_{i0} \, I_0$$

and if the condition that not all δL_{i0} vanish simultaneously is satisfied, then for each motion one has at least one $\delta I_k \sim I_0$. The left-hand side of (17) thus contains factors that are proportional to I_0^2, while the right-hand side is proportional to I_0. If one starts with some current distribution postulated by the equilibrium conditions in a) it is always possible to satisfy (17) with the similarity transformation $I_0, I_{i \geqq 1} \rightarrow A\, I_0, A^{-1} I_{i \geqq 1}$ by making A large enough.

It should be pointed out that the method described here is not the only possible way to achieve stability. As (17) was not a necessary criterion, an attempt could be made to have the second term of (16) exceed the first in magnitude only for such displacements where the first term is negative.

c) *Special Stable Configurations*

Having derived criteria for equilibrium and stability, let us now present concrete examples of stable equilibria. Let the loop L_0 to be stabilized be a circle with its center at the origin of the coordinate system and with its surface normal parallel to the z-axis (see Fig. 2).

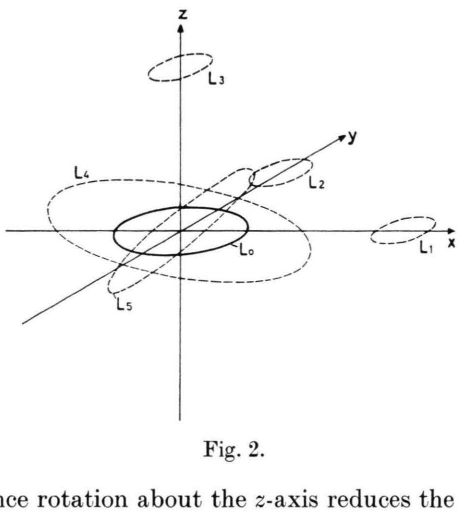

Fig. 2.

Since rotation about the z-axis reduces the circle to itself, only the remaining 5 degrees of freedom are of interest, and according to the method first described in b) another 5 fixed loops $L_1 \ldots L_5$ are required for stabilization. These are also circular and arranged as follows: The centers of L_1, L_2 and L_3 lie a distance a apart on the x, y and z-axis respectively in such a way that their surface normals are parallel to the z-axis.

L_4 and L_5 are symmetric to the origin and tilted 45^0 out of the $z-x$ and $z-y$ planes respectively. The radii of the L_i, $i=1,\ldots,5$ are arbitrary but such that the L_i do not come into contact with one another or with L_0. L_{01}, L_{02} and L_{03} change only on translations of L_0, their spatial gradients becoming perpendicular to one another. On ratations of L_0 about its center on the other hand, they remain unchanged for reasons of symmetry. L_{04} and L_{05} change only on rotations, a maximum being attained on rotation about the y and x-axes respectively. Their (angular) gradients are thus perpendicular to one another. As a result the 5 vectors $\partial L_{i0}/\partial x^\alpha$, $\alpha=1,\ldots 5$; $i=1,\ldots,5$, are orthogonal to one another and therefore satisfy the condition imposed after (20). If we now make just I_0 sufficiently large, stability is guaranteed.

In order to satisfy the equilibrium conditions as well, it is also necessary for at least one of the outer loops to have a non-zero current. Since only the angular gradients of L_4 and L_5 are non-zero and perpendicular to one another, these two loops should, according to (15), not be supplied with current. It is then possible to distribute arbitrary currents $|I_1|+|I_2|+|I_3| \ll |I_0|$ to $L_1 \ldots L_3$ and always satisfy Eq. (14) with a suitable gravitational field (turning the entire system of loops). It is best of all to supply only L_3 with current. This then gives according to (14)

$$I_0\, I_3 \cdot \partial L_{03}/\partial x^\alpha = m_0\, g_\alpha\,, \quad \alpha = 1, 2, 3$$

and since the gradient of L_{03} is in the z-direction, $\boldsymbol{g}$ is in the z-direction (as is required in the levitron). Another advantage is that in the equilibrium position ($\delta I_i = 0$) the entire field is rotational symmetric about the z-axis.

The following example demonstrates how loops without symmetries can also be stabilized: The circular loop chosen above is assumed to be bent arbitrarily. This causes the vectors $\partial L_{i0}/\partial x^\alpha$ (where $\alpha = 1,\ldots,6$) to change in magnitude and direction. For reasons of continuity there is a finite region of bends in which the $\partial L_{i0}/\partial x^\alpha$ still remain linearly independent. To stabilize the new degree of freedom that results from turning L_0 about the z-axis, another loop L_6 has to be arranged in such a way that L_{06} changes most strongly when L_6 is rotated about the z-axis (e.g. forming a circle roughly concentric with L_0 and also slightly bent). All six vectors $\partial L_{i0}/\partial x^\alpha$, $i=1,\ldots,6$ are then linearly independent. As above, only L_3 is supplied with current in addition to L_0. For equilibrium it is also necessary to have

$$I_3 \cdot \partial L_{03}/\partial x^\alpha = 0\,, \quad \alpha = 4, 5, 6\,.$$

This can be achieved by small finite rotations of L_3 and by a — possibly large — rotation of L_0 about the z-axis, the latter rotation being synonymous with a small, continuous bend. (When L_0 is not bent and in the equilibrium position, L_{03} has a maximum with respect to the angle of rotation about the x and y-axes. As a result of the bend the maximum is slightly shifted.) Equation (14) can then be satisfied as before with a suitable gravitational field.

IV. Finite Resistances

If none of the loops L_i has infinite or vanishing resistance, the work (1) expended on displacements ceases to be uniquely integrable. From the law of induction it can be seen, however, that for very slow perturbations ($\dot{L}_{ik} \to 0$ and hence $\dot{\Phi}_i \to 0$) loops with finite resistance behave like loops with infinite resistance $(R_i(I_i - I_i^{(0)}) \to 0\,, \Rightarrow \delta I_i \to 0)$.

For very fast perturbations, on the other hand, they behave like superconductors as follows from (4):

$$\sum_{k=0}^{n} L_{ik}(\Delta \dot{I}_k) + \sum_{k=0}^{n} (R_i \delta_{ik} + \dot{L}_{ik}) \Delta I_k = - \sum_{k=0}^{n} \dot{L}_{ik} I_k^{(0)}$$

with $\Delta I_i = I_i - I_i^{(0)}$. Now, the resistances can be ignored in the second sum, the equation is therefore integrable and after making the appropriate choice of integration constants one obtains

$$\sum_{k=0}^{n} (L_{ik} \Delta I_k + \Delta L_{ik} I_k^{(0)}) = 0$$

or with $L_{ik} = L_{ik}^{(0)} + \Delta L_{ik}$

$$\sum_{k=0}^{n} (L_{ik}^{(0)} \Delta I_k + \Delta L_{ik} I_k^{(0)} + \Delta L_{ik} \Delta I_k) = 0\,.$$

Just after the start of the perturbation the second-order term can be ignored, since the ΔL_{ik} and ΔI_k are still small, and by comparing with (2) one obtains in the lowest order the result $\Delta \Phi_i = 0$.

An arrangement of loops with finite resistance can thus be stabilized with respect to fast perturbations, as was done for superconductors (Section III), but it is nevertheless still unstable to slow perturbations. In any case it is of advantage to keep the resistances as low as possible because the growth rates of the instability are then sure to be small.

It might be supposed that the situation could be improved by means of variable resistances $R_i(I_i)$ (electronic control) and, in particular, that fewer outer loops would be sufficient. Amplifying the induced currents in a configuration stabilized with respect to fast perturbations is sure to improve stability appreciably in the case of slow perturbations. Fewer loops will not suffice in this case, however, since then there will always be motions in which current is not induced in any of the loops, thus preventing automatic amplification.

V. Appendix

In Section I the assumption was introduced that the thickness of the loops is small relative to the distances between them and to their longitudinal dimensions. In this model it was plausible to regard the self-inductances L_{ii} as constant and finite quantities. Nevertheless the loops are of finite diameter, and so certain changes in the current distribution do result from motions, and hence the self-inductances vary as well. If, however, the current paths are fixed by making the loops infinitely thin, the self-inductances tend to infinity.

The extent to which this affects the stability problem forms the subject of this appendix.

Attention is confined to superconductors and the case which was studied in section III

$$|I_i| \ll |I_0|\,, \quad i = 1, \dots, n\,.$$

Allowing now for the variability of the self-inductances it follows with (18) from the constancy of the fluxes that

$$\delta I_i = - \sum_{k,j=0}^{n} L_{ik}^{-1} \delta L_{kj} I_j \approx - \sum_{k=0}^{n} L_{ik}^{-1} \delta L_{k0} I_0 \quad \text{(A1)}$$

$$(L_{ik}^{-1} = \text{inverse matrix to } L_{ik}).$$

Therefore, according to (12) and (A1) one obtains the approximation

$$\delta^{(2)} W = - \tfrac{1}{2} \sum_{i,k=0}^{n} I_i I_k \delta^{(2)} L_{ik} + \sum_{i,k=0}^{n} \delta I_i \delta I_k L_{ik}$$
$$\approx I_0^2 \{ - \tfrac{1}{2} \delta^{(2)} L_{00} + \sum_{i,k=0}^{n} \delta L_{0i} \delta L_{0k} L_{ik}^{-1} \}. \quad \text{(A2)}$$

Postulating large distances between the loops means that

$$|(L_{ik} - \delta_{ik} L_{ii})/L_{jj}| \ll 1\,. \quad \text{(A3)}$$

Setting

$$L_{ik}(\varepsilon) = L_{ii} \delta_{ik} + \varepsilon (L_{ik} - \delta_{ik} L_{ii})$$

one obtains the elements of the inverse matrix approximately from the series expansion

$$L_{km}^{-1}(\varepsilon) = l_{km}^{(0)} + \varepsilon\, l_{km}^{(1)} + o(\varepsilon)$$

by successively solving the equation

$$\sum_k L_{ik}(\varepsilon)\, L_{km}^{-1}(\varepsilon) = \delta_{ik}$$

in ascending orders of ε and then setting $\varepsilon = 1$. This gives to the first order

$$L_{km}^{-1} \approx (1/L_{kk}) \{\delta_{km} - (L_{km} - \delta_{km} L_{kk})/L_{mm}\},$$

and substituting this in (A2) one obtains

$$\delta^{(2)} W \approx I_0^2 \left\{ -\frac{1}{2}\, \delta^{(2)} L_{00} + \sum_{k=0}^{n} \frac{(\delta L_{0k})^2}{L_{kk}} - \sum_{k,m=0}^{n} \frac{\delta L_{0k}\, \delta L_{0m}}{L_{kk} L_{mm}} (L_{km} - \delta_{km} L_{kk}) \right\}.$$

Since the second term in the bracket was arranged in Section III such that it does not vanish anywhere, $\delta^{(2)} W$ can also be written

$$\delta^{(2)} W \approx I_0^2 \left\{ -\frac{1}{2}\, \delta^{(2)} L_{00} + \sum_{k=0}^{n} \frac{(\delta L_{0k})^2}{L_{kk}} \right\} = I_0^2 \left\{ -\frac{1}{2}\, \delta^{(2)} L_{00} + \frac{(\delta L_{00})^2}{L_{00}} + \sum_{k=1}^{n} \frac{(\delta L_{0k})^2}{L_{kk}} \right\} \qquad (A4)$$

because of (A3). All terms here are positive except the first, its sign being difficult to predict. (To do this one would really have to solve the potential problem for finite loops.) If it is negative, there are three possible ways of keeping it so small as to prevent destabilizing effects.

1) First one could try to arrange the experimental situation such that

$$(\delta^{(2)} L_{00})/L_{00} < 2\, (\delta L_{00})^2/L_{00}\,.$$

Variation of L_0 in itself would cause stabilization.

2) The second approach, which is more feasible experimentally would be to try obtain the inequality

$$|\delta^{(2)} L_{00}| < (2/L_{ii})\, (\delta L_{0i})^2$$

for at least one $i \neq 0$. For this purpose it would be of advantage to give the outer loops small self-inductances and make L_{00} as weakly variable as possible. Since variations of L_{00} are caused by non-uniform changes of the current distribution in L_0, one could segment the surface of L_0 in the longitudinal direction or wind a thin, superconducting wire many times along the surface parallel to the axis. This produces a quasi-continuous current distribution which, however, is fixed in the longitudinal direction.

3) The third possibility would be to increase the (positive) summation term in (A4) by adding more outer loops.

VI. Conclusions

Whereas a loop with constant current cannot be stably levitated in the field of similar fixed loops, it was demonstrated in Section III that it is possible in principle to levitate a superconductor by means of other fixed superconductors. For this purpose at least 6 fixed outer loops were generally needed, it being of advantage that only one of them had to have a current. If the loop to be levitated is circular, the current-carrying outer loop can be placed in such a way that the total field of both loops in equilibrium has rotational symmetry. Since the main concern in the levitron, for example, is the magnetic field of the free loop, the configuration with $I_i \ll I_0$ (current in the outer loop much smaller than in the free loop) that is described in Section III c proves to be particularly favourable. The rotational symmetry of the field is perturbed only when the free loop oscillates about the equilibrium position, such perturbations being small relative to the field of the free loop.

If the free loop does not have to be levitated for too long a period, it is sufficient for practical purposes to provide the outer loops with a finite but small resistance. In this case the stability can be improved by electronically readjusting the resistances to amplify the induced currents.

It can be assumed that the possibly destabilizing effects discussed in the appendix that result from a change in the self-inductance L_{00} can either be neglected or kept sufficiently small in the ways suggested in the appendix.

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Prof. Schlüter has shown up to second order that the impossibility of stabilizing loops with constant current (proved in Section II) is generally valid for two continuous current distributions that are moved rigidly relative to one another without coming into contact. This proof was the starting point for this work. The authors would like to thank him and also Dr. Lortz, Dr. Wimmel and Dr. Wobig for valuable discussions.

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